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Integrable N-particle Hamiltonians with Yangian or reflection algebra symmetry

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Abstract
We use the Dunkl operator approach to construct one-dimensional integrable models describing N particles with internal degrees of freedom. These models are described by a general Hamiltonian belonging to the centre of the Yangian or the reflection algebra, which ensures that they admit the corresponding symmetry. In particular, the open problem of the symmetry is answered for the B_N-type Sutherland model with spin and for a generalized B_N-type nonlinear Schrödinger Hamiltonian.

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Introduction

The introduction of internal degrees of freedom in an increasing number of one-dimensional quantum integrable systems has proved to be fruitful in various physical and mathematical investigations. This is well illustrated in the study of symmetries. In particular, the Yangian symmetry was exhibited in the A_N Sutherland model with spin [1], the A_N confined Calogero model with spin [2] or the quantum nonlinear Schrödinger (NLS) equation [3, 4]. This in turn allows us to find the spectrum and degeneracies.

The main idea of this paper is to generalize the Dunkl operator approach of [1] in order to construct a general N-body Hamiltonian which possesses the reflection algebra [5] as symmetry algebra. A direct consequence is the integrability of the system described by this general Hamiltonian. Taking a particular case, we answer the question of the symmetry of the B_N Sutherland model with spin. In the same way, we exhibit the symmetry of a generalized B_N-type NLS Hamiltonian. With the same procedure, we also construct a general N-body integrable Hamiltonian with Yangian symmetry from which we recover the known cases of NLS and A_N Sutherland model with spin.

After recalling some known mathematical background needed in the construction of the central elements of the Yangian [6] of $gl(n)$, $Y(n)$, and of the reflection algebra, $\mathcal{B}(n)$, in section 1, we give a realization of these algebras in terms of transfer matrices and generators of the *extended* degenerate affine Hecke algebra, $\mathcal{A}(N)$. Next, we prove the main theorems of section 2 which provide another realization for each algebra $\mathcal{B}(n)$ and $Y(n)$ in terms of a projector specifying the physical properties of the wavefunctions occurring when we represent our setup in section 3. We identify a central element used in section 3 (resp. section 4) to construct the general one-dimensional N-particle Hamiltonian for which we prove integrability and reflection algebra (resp. Yangian) symmetry. This is done by representing $\mathcal{A}(N)$ in terms of operators (in particular Dunkl operators) acting on the space of wavefunctions. Then, we particularize the former general Hamiltonian and conclude on the symmetry of generalizations of NLS and Sutherland models.

1. Central elements of $Y(n)$ and $\mathcal{B}(n)$

We deal with the multiple tensor products $(\mathrm{End}(\mathbb{C}^n))^{\otimes m}$ where $m \in \mathbb{Z}_{\geqslant 0}$ will be the number of copies necessary for the equations to make sense. For $A \in \mathrm{End}(\mathbb{C}^n)$ and $k \in \{1, \ldots, m\}$, we define A_k by

$$A_k = 1^{\otimes k-1} \otimes A \otimes 1^{\otimes m-k} \in (\mathrm{End}(\mathbb{C}^n))^{\otimes m}. \tag{1.1}$$

1.1. Yangian $Y(n)$

The Yangian of gl_n [6], $Y(n)$, is the complex associative algebra, generated by the unit and the elements $\left\{t_{ij}^{(k)} \middle| 1 \leqslant i, j \leqslant n; k \in \mathbb{Z}_{>0}\right\}$ gathered in the formal series

$$t_{ij}(u) = \delta_{ij} + \lambda \sum_{k \in \mathbb{Z}_{>0}} t_{ij}^{(k)} u^{-k} \in Y(n)[[u^{-1}]] \tag{1.2}$$

subject to the defining relations

$$(u - v)[t_{ij}(u), t_{kl}(v)] = \lambda(t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u)) \tag{1.3}$$

where λ is the parameter of deformation of the Yangian. Let E_{ij} be the elementary matrix with entry 1 in row i and column j and zero elsewhere and $T(u)$ be defined by

$$T(u) = \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij} \in Y(n)[[u^{-1}]] \otimes \mathrm{End}(\mathbb{C}^n). \tag{1.4}$$

Then the relations (1.3) are equivalent to the RTT relation [7]

$$R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v) \tag{1.5}$$

where

$$R_{12}(u) = 1 \otimes 1 - \lambda \frac{P_{12}}{u} \qquad P_{12} = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n). \tag{1.6}$$

P_{12} is the permutation operator, i.e. $P_{12} v \otimes w = w \otimes v$, with $v, w \in \mathbb{C}^n$.

This R-matrix, called the Yang matrix, satisfies the following properties:

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v) \qquad \text{(Yang–Baxter equation)} \tag{1.7}$$

$$R_{12}(u)R_{12}(-u) = \frac{u^2 - \lambda^2}{u^2} 1 \otimes 1 \qquad \text{(unitarity relation)}. \tag{1.8}$$

Let A_m be the antisymmetrizer operator in $(\mathbb{C}^n)^{\otimes m}$ i.e.

$$A_m(e_{i_1} \otimes \cdots \otimes e_{i_m}) = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(m)}} \tag{1.9}$$

where $\{e_i | 1 \leqslant i \leqslant n\}$ is the canonical basis of $\mathbb{C}^n$ and $1 \leqslant i_1, \ldots, i_m \leqslant n$. One can show (see, e.g., [8]) that the following identities hold:

$$A_m T_1(u) \cdots T_m(u - m\lambda + \lambda) = T_m(u - m\lambda + \lambda) \cdots T_1(u) A_m. \tag{1.10}$$

For $m = n$, A_n becomes a one-dimensional operator in $(\mathbb{C}^n)^{\otimes n}$ and the element (1.10) is then equal to A_n times a scalar series with coefficients in $Y(n)$ called the quantum determinant. This reads

$$A_n \,\mathrm{qdet}\, T(u) = A_n T_1(u) \cdots T_n(u - n\lambda + \lambda). \tag{1.11}$$

A well-known result (see, e.g., [9]) is that the coefficients of qdet $T(u)$ generate the centre of $Y(n)$.

1.2. Reflection algebra $\mathcal{B}(n)$

Let $Q \in \mathrm{End}(\mathbb{C}^n)$ be an operator such that $Q^2 = 1$. Let us introduce $\widetilde{\mathcal{B}}(n)$ the complex associative algebra generated by the unit and the elements $\{\tilde{s}_{ij}^{(k)} | 1 \leqslant i, j \leqslant n; k \in \mathbb{Z}_{\geqslant 0}\}$ gathered in the formal series

$$\tilde{s}_{ij}(u) = \sum_{k \in \mathbb{Z}_{\geqslant 0}} \tilde{s}_{ij}^{(k)} u^{-k} \in \widetilde{\mathcal{B}}(n)[[u^{-1}]]. \tag{1.12}$$

The defining relations are given by the reflection equation [5, 10]

$$R_{12}(u - v) \widetilde{S}_1(u) Q_1 R_{12}(u + v) Q_1 \widetilde{S}_2(v) = \widetilde{S}_2(v) Q_1 R_{12}(u + v) Q_1 \widetilde{S}_1(u) R_{12}(u - v) \tag{1.13}$$

where

$$\widetilde{S}(u) = \sum_{i,j=1}^{n} \tilde{s}_{ij}(u) \otimes E_{ij} \in \widetilde{\mathcal{B}}(n)[[u^{-1}]] \otimes \mathrm{End}(\mathbb{C}^n). \tag{1.14}$$

There exists a connection between $Y(n)$ and $\widetilde{\mathcal{B}}(n)$.

Theorem 1.1. [5] *Let*

$$B(u) = \sum_{k \geqslant 0} \frac{B^{(k)}}{u^k} \in \mathrm{End}(\mathbb{C}^n)[[u^{-1}]]$$

satisfy the relation (1.13). Then, the map

$$\begin{aligned} \phi : \widetilde{\mathcal{B}}(n) &\longmapsto Y(n) \\ \widetilde{S}(u) &\longrightarrow S(u) \equiv T(u) B(u) Q T^{-1}(-u) Q \end{aligned} \tag{1.15}$$

defines an algebra homomorphism.

In this paper, we consider the reflection algebra $\mathcal{B}(n)$, subalgebra of $Y(n)$, defined as the image of $\widetilde{\mathcal{B}}(n)$ by ϕ.

By the same procedure as in [11], one can define the Sklyanin determinant

$$\begin{aligned} A_n \,\mathrm{sdet}\, S(u) = A_n \prod_{1 \leqslant k \leqslant n-1}^{\longrightarrow} &(S_k(u + \lambda - k\lambda) R_{k,k+1} \\ &\times (2u + \lambda(1 - 2k)) \cdots R_{k,n}(2u + \lambda(2 - k - n))) S_n(u + \lambda - n\lambda) \end{aligned} \tag{1.16}$$

where the product is ordered i.e. $\overrightarrow{\prod}_{1\leqslant k\leqslant n-1} X_k = X_1 \cdots X_{n-1}$. Following [11], one can express the Sklyanin determinant in terms of the quantum determinant

$$\operatorname{sdet} S(u) = \theta(u) \operatorname{qdet} T(u) (\operatorname{qdet} T(-u + n\lambda - \lambda))^{-1} \tag{1.17}$$

where $\theta(u) = \operatorname{sdet} B(u) \in \mathbb{C}[[u^{-1}]]$.

From theorem 1.1 and relation (1.17), one deduces that the coefficients of the Sklyanin determinant belong to the centre of $\mathcal{B}(n)$, which will be fundamental in establishing the reflection symmetry.

2. Realizations of $Y(n)$ and $\mathcal{B}(n)$

This section is the first step towards our goal. By realizing the above algebras, we will identify what will be interpreted as Hamiltonians in the following sections.

2.1. Extended degenerate affine Hecke algebra

Let $N \in \mathbb{Z}_{\geqslant 2}$. The extended degenerate affine Hecke algebra, $\mathcal{A}(N)$, is the complex associative algebra generated by the unit and three sets of elements denoted $\{d_i | 1 \leqslant i \leqslant N\}$, $\{\mathcal{P}_{i,i+1} | 1 \leqslant i \leqslant N-1\}$ and $\{\mathcal{Q}_i | 1 \leqslant i \leqslant N\}$ subject to the defining relations

$$\mathcal{P}_{i,i+1}\mathcal{P}_{i+1,i+2}\mathcal{P}_{i,i+1} = \mathcal{P}_{i+1,i+2}\mathcal{P}_{i,i+1}\mathcal{P}_{i+1,i+2} \tag{2.1}$$

$$\mathcal{P}_{i,i+1}^2 = 1 \tag{2.2}$$

$$\mathcal{P}_{i,i+1} d_k = \begin{cases} d_k \mathcal{P}_{i,i+1} & k \neq i, i+1 \\ d_{i+1}\mathcal{P}_{i,i+1} + \beta & k = i \\ d_i \mathcal{P}_{i,i+1} - \beta & k = i+1 \end{cases} \tag{2.3}$$

$$[d_i, d_j] = 0 \tag{2.4}$$

$$\mathcal{Q}_i^2 = 1 \tag{2.5}$$

$$\mathcal{Q}_i \mathcal{Q}_j = \mathcal{Q}_j \mathcal{Q}_i \tag{2.6}$$

$$\mathcal{Q}_i \mathcal{P}_{k,k+1} = \begin{cases} \mathcal{P}_{k,k+1}\mathcal{Q}_i & i \neq k, k+1 \\ \mathcal{P}_{k,k+1}\mathcal{Q}_{k+1} & i = k \\ \mathcal{P}_{k,k+1}\mathcal{Q}_k & i = k+1 \end{cases} \tag{2.7}$$

$$\mathcal{Q}_i d_k = \begin{cases} d_k \mathcal{Q}_i & k < i \\ -d_i \mathcal{Q}_i + \beta \sum_{j=i+1}^N \mathcal{P}_{ij}(\mathcal{Q}_i + \mathcal{Q}_j) + b & k = i \\ d_k \mathcal{Q}_i + \beta \mathcal{P}_{ik}(\mathcal{Q}_i - \mathcal{Q}_k) & k > i \end{cases} \tag{2.8}$$

where

$$\beta \in \mathbb{C} \qquad b \in \mathbb{C} \tag{2.9}$$

and

$$\mathcal{P}_{ij} = \mathcal{P}_{i,i+1}\mathcal{P}_{i+1,i+2}\cdots\mathcal{P}_{j-2,j-1}\mathcal{P}_{j-1,j}\mathcal{P}_{j-2,j-1}\cdots\mathcal{P}_{i+1,i+2}\mathcal{P}_{i,i+1}. \tag{2.10}$$

The commutation relations (2.1)–(2.8) were obtained in [12] for a particular representation but here we set them as abstract algebraic relations.

Let us note that the subalgebra of $\mathcal{A}(N)$ generated by $\{d_i | i = 1, \ldots, N\}$ and $\{\mathcal{P}_{i,i+1} | i = 1, \ldots, N-1\}$ satisfying relations (2.1)–(2.4) is the degenerate affine Hecke algebra denoted $\widetilde{\mathcal{A}}(N)$ first introduced in [13].

2.2. Transfer matrix

In order to realize $Y(n)$ and $\mathcal{B}(n)$ in terms of the elements of $\mathcal{A}(N)$, we suppose that the latter commute with P and Q. A realization of $Y(n)$ is given by the transfer matrix [1]

$$\mathcal{T}_0(u) = \mathcal{L}_{01}(u)\cdots\mathcal{L}_{0N}(u) \in \mathrm{End}(\mathbb{C}^n)\otimes\mathrm{End}(\mathbb{C}^n)^{\otimes N} \tag{2.11}$$

where

$$\mathcal{L}_{0i}(u) = \frac{u+d_i}{u+d_i-\lambda}R_{0i}(u+d_i) = \frac{u+d_i}{u+d_i-\lambda}\left(1-\frac{\lambda P_{0i}}{u+d_i}\right). \tag{2.12}$$

The first space denoted 0 in (2.11) is called the auxiliary space. The other ones, denoted $1,\ldots,N$ and not displayed explicitly in $\mathcal{T}_0(u)$ for brevity, are called the quantum spaces.

In the realization (2.11) of $Y(n)$, the quantum determinant takes the following particular form:

$$\mathrm{qdet}\,\mathcal{T}(u) = \prod_{j=1}^{N}\frac{u+d_j}{u+d_j-n\lambda+\lambda}. \tag{2.13}$$

This realization allows us to obtain a realization of $\mathcal{B}(n)$ thanks to theorem 1.1 and relation (1.8)

$$\mathcal{S}_0(u) = \mathcal{T}_0(u)B_0(u)Q_0\mathcal{T}_0^{-1}(-u)Q_0 \tag{2.14}$$

$$\begin{aligned}= &\frac{u+d_1}{u+d_1-\lambda}\left(1-\frac{\lambda P_{01}}{u+d_1}\right)\cdots\frac{u+d_N}{u+d_N-\lambda}\left(1-\frac{\lambda P_{0N}}{u+d_N}\right)B_0(u)Q_0\\ &\times\frac{u-d_N}{u-d_N-\lambda}\left(1-\frac{\lambda P_{0N}}{u-d_N}\right)\cdots\frac{u-d_1}{u-d_1-\lambda}\left(1-\frac{\lambda P_{01}}{u-d_1}\right)Q_0\end{aligned} \tag{2.15}$$

and one can compute

$$\mathrm{sdet}\,\mathcal{S}(u) = \theta(u)\prod_{j=1}^{N}\frac{(u+d_j)(-u+d_j)}{(u+d_j-n\lambda+\lambda)(-u+d_j+n\lambda-\lambda)} \tag{2.16}$$

$$\begin{aligned}&= \theta_0+\frac{1}{u}(\theta_1+2(n\lambda-\lambda)N\theta_0)+\frac{1}{u^2}(\theta_2+2(n\lambda-\lambda)N\theta_1+(n\lambda-\lambda)^2N(2N+1)\theta_0)\\ &\quad+\frac{1}{u^3}\Bigg(\theta_3+2(n\lambda-\lambda)N\theta_2+(n\lambda-\lambda)^2N(2N+1)\theta_1\\ &\quad+(n\lambda-\lambda)^3\frac{2N(N+1)(2N+1)}{3}\theta_0+2\theta_0\mathcal{H}\Bigg)+O\left(\frac{1}{u^4}\right)\end{aligned} \tag{2.17}$$

where

$$\mathcal{H} = \sum_{i=1}^{N}d_i^2 \tag{2.18}$$

and the coefficients θ_j $(j = 0, 1, 2, 3)$ are given by the expansion

$$\theta(u) = \mathrm{sdet}\,B(u) = \theta_0+\frac{\theta_1}{u}+\frac{\theta_2}{u^2}+\frac{\theta_3}{u^3}+O\left(\frac{1}{u^4}\right). \tag{2.19}$$

As announced earlier, we identified a central element $\mathcal{H}$ whose interpretation as Hamiltonian will become explicit in sections 3 and 4.

2.3. Projectors

We now turn to a crucial point in our construction. Let us define two operators

$$\Lambda^{(1)} = \frac{1}{N!}\prod_{j=2}^{N}(1 + \tau' P_{1j}\mathcal{P}_{1j} + \cdots + \tau' P_{j-1,j}\mathcal{P}_{j-1,j}) \tag{2.20}$$

$$\Lambda^{(2)} = \frac{1}{2^N}\prod_{j=1}^{N}(1 + \tau'' Q_j \mathcal{Q}_j) \tag{2.21}$$

where $\tau', \tau'' = \pm 1$. We define $\Lambda = \Lambda^{(1)}\Lambda^{(2)} = \Lambda^{(2)}\Lambda^{(1)}$. One can check that the operators $\Lambda^{(1)}$, $\Lambda^{(2)}$ and Λ are projectors. Let us remark that the products in relations (2.20) and (2.21) are not necessarily ordered since the factors in each product commute with one another.

Lemma 2.1. *For $1 \leqslant i < j \leqslant N$ and $1 \leqslant l \leqslant N$, one has*

$$(1 - \tau' P_{ij}\mathcal{P}_{ij})\Lambda^{(1)} = 0 \tag{2.22}$$

$$(1 - \tau'' Q_l \mathcal{Q}_l)\Lambda^{(2)} = 0. \tag{2.23}$$

Proof. Let $\sigma \in \mathfrak{S}_N$. An equivalent definition of $\Lambda^{(1)}$ is

$$\Lambda^{(1)} = \frac{1}{N!}\prod_{k=2}^{N}(1 + \tau' P_{\sigma(1)\sigma(k)}\mathcal{P}_{\sigma(1)\sigma(k)} + \cdots + \tau' P_{\sigma(k-1)\sigma(k)}\mathcal{P}_{\sigma(k-1)\sigma(k)}).$$

For $1 \leqslant i < j \leqslant N$, let us choose σ so that $\sigma(1) = i$ and $\sigma(2) = j$. Then, one obtains

$$(1 - \tau'\mathcal{P}_{ij}P_{ij})\Lambda^{(1)} = (1 - \tau'\mathcal{P}_{ij}P_{ij})(1 + \tau'\mathcal{P}_{ij}P_{ij})$$
$$\times \frac{1}{N!}\prod_{k=3}^{N}(1 + \tau' P_{\sigma(1)\sigma(k)}\mathcal{P}_{\sigma(1)\sigma(k)} + \cdots + \tau' P_{\sigma(k-1)\sigma(k)}\mathcal{P}_{\sigma(k-1)\sigma(k)}) = 0 \tag{2.24}$$

which proves relation (2.22). Relation (2.23) is straightforward. □

In the rest of this paper, we take a particular form for $B(u)$

$$B(u) = 1 + b'\frac{Q}{u} \qquad (b' \in \mathbb{C}). \tag{2.25}$$

In this case, the constant coefficient θ_0 in (2.19) is 1. Let us now state the main theorem of this section.

Theorem 2.2. *If $\beta = \tau'\lambda$ and $b = -2\tau'' b'$, then $\mathcal{S}(u)\Lambda$ is a realization of $\mathcal{B}(n)$ i.e. one obtains*

$$R_{00'}(u-v)\mathcal{S}_0(u)\Lambda Q_0 R_{00'}(u+v)Q_0\mathcal{S}_{0'}(v)\Lambda = \mathcal{S}_{0'}(v)\Lambda Q_0 R_{00'}(u+v)Q_0\mathcal{S}_0(u)\Lambda R_{00'}(u-v). \tag{2.26}$$

The Sklyanin determinant can be computed thanks to the following formula:

$$\mathrm{sdet}(\mathcal{S}(u)\Lambda) = (\mathrm{sdet}\,\mathcal{S}(u))\Lambda. \tag{2.27}$$

Proof. Noting that Λ commutes with $R_{00'}$ and Q_0, the validity of relation (2.26) is implied by

$$(\Lambda - 1)\mathcal{S}_0(u)\Lambda = 0. \tag{2.28}$$

This in turn holds if

$$\begin{cases}(\mathcal{P}_{i,i+1} - \tau' P_{i,i+1})\mathcal{S}_0(u)\Lambda^{(1)} = 0 \qquad i = 1, \ldots, N-1\\ (\mathcal{Q}_N - \tau'' Q_N)\mathcal{S}_0(u)\Lambda^{(2)} = 0.\end{cases} \tag{2.29}$$

Now a direct computation using the exchange relations of $\mathcal{A}(N)$ and the conditions on β and b allows one to find $\mathcal{S}'$ and $\mathcal{S}''$ such that

$$\begin{cases} (\mathcal{P}_{i,i+1} - \tau' P_{i,i+1})\mathcal{S}_0(u) = \mathcal{S}'_0(u)(\mathcal{P}_{i,i+1} - \tau' P_{i,i+1}) \\ (\mathcal{Q}_N - \tau'' Q_N)\mathcal{S}_0(u) = \mathcal{S}''_0(u)(\mathcal{Q}_N - \tau'' Q_N) \end{cases} \tag{2.30}$$

which finishes the proof of (2.28) invoking lemma 2.1. Relation (2.27) is proved using the definition (1.15) of the Sklyanin determinant and relation (2.28). □

Remark. One can verify that the validity of (2.26) actually imposes the explicit form (2.25) of $B(u)$ up to a normalization and the above constraints on λ and b'.

In a similar way, one can prove the following theorem. The latter encompasses the analogue result in [1]. Indeed, one recovers the situation of [1] by specifying a particular representation of the generators of $\mathcal{A}(N)$.

Theorem 2.3. *If $\beta = \tau'\lambda$, then $\mathcal{T}(u)\Lambda^{(1)}$ is a realization of $Y(n)$ i.e. one obtains*

$$R_{00'}(u-v)\mathcal{T}_0(u)\Lambda^{(1)}\mathcal{T}_{0'}(v)\Lambda^{(1)} = \mathcal{T}_{0'}(v)\Lambda^{(1)}\mathcal{T}_0(u)\Lambda^{(1)}R_{00'}(u-v). \tag{2.31}$$

The quantum determinant of $\mathcal{T}(u)\Lambda^{(1)}$ can be computed thanks to the following formula:

$$\mathrm{qdet}(\mathcal{T}(u)\Lambda^{(1)}) = (\mathrm{qdet}\,\mathcal{T}(u))\Lambda^{(1)} \tag{2.32}$$

Proof. The proof is similar to that of theorem 2.2. □

3. Hamiltonians with $\mathcal{B}(n)$ symmetry

In this section and the following one, we present the physical application of the above mathematical setting. We will work in the first quantized picture with N indistinguishable particles. Let $\{q_i|1 \leqslant i \leqslant N\}$ be the coordinates and $\{s_i|1 \leqslant i \leqslant N\}$ the internal degrees of freedom (or spins) of the particles. Any s_i takes values in $\Sigma = \left\{-\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2}\right\}$. Then, the wavefunction of the system is denoted $\phi(q_1, \ldots, q_N|s_1, \ldots, s_N)$.

3.1. Representation of $\mathcal{A}(N)$ and associated Hamiltonians

We represent P, Q and the generators of $\mathcal{A}(N)$ as operators on the space $\mathfrak{L}$ of wavefunctions. This reads, for $1 \leqslant i < j \leqslant N$ and $\phi \in \mathfrak{L}$,

$$\mathcal{P}_{ij}\phi(q_1, \ldots, q_i, \ldots, q_j, \ldots, q_N|s_1, \ldots, s_N) = \phi(q_1, \ldots, q_j, \ldots, q_i, \ldots, q_N|s_1, \ldots, s_N) \tag{3.1}$$

$$P_{ij}\phi(q_1, \ldots, q_N|s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N) = \phi(q_1, \ldots, q_N|s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N) \tag{3.2}$$

i.e. $\mathcal{P}_{ij}$ (resp. P_{ij}) is the permutation operator acting on positions (resp. spins) of the ith and jth particles. And for $1 \leqslant i \leqslant N$, we define

$$\mathcal{Q}_i\phi(q_1, \ldots, q_i, \ldots, q_N|s_1, \ldots, s_N) = \phi(q_1, \ldots, \alpha(q_i), \ldots, q_N|s_1, \ldots, s_N) \tag{3.3}$$

$$Q_i\phi(q_1, \ldots, q_N|s_1, \ldots, s_i, \ldots, s_N) = \phi(q_1, \ldots, q_N|s_1, \ldots, s_i^*, \ldots, s_N) \tag{3.4}$$

where α is a function defining the action of $\mathcal{Q}_i$ on the position of the ith particle and $*$ represents the action of Q_i on its spin. Since $\mathcal{Q}_i^2 = 1$ and $Q_i^2 = 1$, one obtains $\alpha(\alpha(q_i)) = q_i$ and $(s_i^*)^* = s_i$. Now, we choose d_l to be a Dunkl operator [14] defined as follows, for $1 \leqslant l \leqslant N$,

$$d_l = a(q_l)\frac{\partial}{\partial q_l} + \sum_{k=1}^{l-1} v(q_l, q_k)\mathcal{P}_{kl} - \sum_{k=l+1}^{N} v(q_k, q_l)\mathcal{P}_{lk} + \sum_{k=1,k\neq l}^{N} \overline{v}(q_l, q_k)\overline{\mathcal{P}}_{lk} + g(q_l)\mathcal{Q}_l \qquad (3.5)$$

where $\overline{\mathcal{P}}_{lk} = \mathcal{Q}_l\mathcal{Q}_k\mathcal{P}_{lk}$. For the product of Dunkl operators to be well defined, $a, v, \overline{v}, g$ must be C^∞ functions.

Theorem 3.1. *For $a \neq 0$ and $A(x) = \int^x \frac{\mathrm{d}y}{a(y)}$ invertible, the operators $\mathcal{P}_{ij}$, $\mathcal{Q}_i$ and d_i as defined in (3.1), (3.3) and (3.5) realize $\mathcal{A}(N)$ if and only if*

$$\alpha(x) = A^{-1}(-A(x)) \qquad (3.6)$$

$$v(x, y) = \frac{\beta}{\mathrm{e}^{-2\gamma(A(x)-A(y))} - 1} \qquad \gamma \in \mathbb{C} \qquad (3.7)$$

$$\overline{v}(x, y) = \frac{\beta}{1 - \mathrm{e}^{2\gamma(A(x)+A(y))}} \qquad (3.8)$$

$$g(x) = \frac{c - b\,\mathrm{e}^{-2\gamma A(x)}}{2\sinh(2\gamma A(x))} \qquad c \in \mathbb{C}. \qquad (3.9)$$

Proof. The constraints on the functions α, a, $v, \overline{v}$ and g arise from (2.3), (2.4) and (2.8). Starting from (2.4), the idea is to cancel the coefficients appearing in front of independent operators such as $\mathcal{P}_{ij}$ or $\mathcal{P}_{ik}\mathcal{P}_{jk}$:

$$a(x)\frac{\partial}{\partial x}v(x, y) + a(y)\frac{\partial}{\partial y}v(x, y) = 0 \qquad (3.10)$$

$$-v(y, z)v(x, z) + v(x, y)v(y, z) + v(x, z)v(y, x) = 0 \qquad (3.11)$$

whose solution is given by

$$v(x, y) = \frac{C}{\mathrm{e}^{-2\gamma(A(x)-A(y))} - 1} \qquad C, \gamma \in \mathbb{C}$$

and (2.3) imposes $C = \beta$. The form of $\alpha, \overline{v}$ and g are found in the same way. Then, a global check ensures that all the remaining relations are identically satisfied. □

The Dunkl operators realized as in (3.5) are independent and from (2.4), (2.18), we have

$$[\mathcal{H}, d_i] = 0 \qquad \text{for} \quad i = 1, \ldots, N \qquad (3.12)$$

which ensures the integrability. Then, from (2.18), we can compute

$$\begin{aligned}\mathcal{H} = &\sum_{i=1}^{N}\left(a(q_i)^2\frac{\partial^2}{\partial q_i^2} + a(q_i)\frac{\partial a(q_i)}{\partial q_i}\frac{\partial}{\partial q_i}\right)\\ &+ \sum_{1\leqslant i<j\leqslant N}\left(\frac{\beta\gamma\left(\mathcal{P}_{ij} - \frac{\beta}{2\gamma}\right)}{\sinh^2[\gamma(A(q_i) - A(q_j))]} + \frac{\beta\gamma\left(\overline{\mathcal{P}}_{ij} - \frac{\beta}{2\gamma}\right)}{\sinh^2[\gamma(A(q_i) + A(q_j))]}\right)\\ &+ \sum_{i=1}^{N}\left(\frac{\gamma(b+c)\left(\mathcal{Q}_i - \frac{b+c}{4\gamma}\right)}{4\sinh^2[\gamma A(q_i)]} - \frac{\gamma(b-c)\left(\mathcal{Q}_i - \frac{b-c}{4\gamma}\right)}{4\cosh^2[\gamma A(q_i)]}\right).\end{aligned} \qquad (3.13)$$

Now the constructions of the previous sections get their physical meaning. $\Lambda^{(1)}$ is the projector from $\mathfrak{L}$ onto $\mathfrak{L}^{(1)}_{\tau'}$, the space of globally τ'-symmetric wavefunctions ($\tau' = 1$ for symmetric and

$\tau' = -1$ for antisymmetric). $\Lambda^{(2)}$ is the projector from $\mathfrak{L}$ onto $\mathfrak{L}^{(2)}_{\tau''}$, the space of wavefunctions such that

$$\phi(q_1, \ldots, \alpha(q_i), \ldots, q_N | s_1, \ldots, s_i^*, \ldots, s_N) = \tau''\phi(q_1, \ldots, q_i, \ldots, q_N | s_1, \ldots, s_i, \ldots, s_N). \tag{3.14}$$

And then, Λ is the projector from $\mathfrak{L}$ onto $\mathfrak{L}_\Lambda = \mathfrak{L}^{(1)}_{\tau'} \cap \mathfrak{L}^{(2)}_{\tau''}$.

Theorem 3.2. *Let $\overline{P}_{ij} = Q_i Q_j P_{ij}$ and $c' = -\frac{c\tau''}{2}$. The effective Hamiltonian, $\mathcal{H}_\Lambda$, restricted to $\mathfrak{L}_\Lambda$, reads*

$$\begin{aligned}
\mathcal{H}_\Lambda = &\sum_{i=1}^{N} \left(a(q_i)^2 \frac{\partial^2}{\partial q_i^2} + a(q_i) \frac{\partial a(q_i)}{\partial q_i} \frac{\partial}{\partial q_i} \right) \\
&+ \sum_{1\leqslant i<j\leqslant N} \left(\frac{\gamma\lambda\left(P_{ij} - \frac{\lambda}{2\gamma}\right)}{\sinh^2[\gamma(A(q_i) - A(q_j))]} + \frac{\gamma\lambda\left(\overline{P}_{ij} - \frac{\lambda}{2\gamma}\right)}{\sinh^2[\gamma(A(q_i) + A(q_j))]} \right) \\
&+ \sum_{i=1}^{N} \left(-\frac{\gamma(b'+c')\left(Q_i + \frac{b'+c'}{2\gamma}\right)}{2\sinh^2[\gamma A(q_i)]} + \frac{\gamma(b'-c')\left(Q_i + \frac{b'-c'}{2\gamma}\right)}{2\cosh^2[\gamma A(q_i)]} \right)
\end{aligned} \tag{3.15}$$

and admits the reflection algebra as symmetry algebra. This ensures in particular that it is integrable.

Proof. $\mathcal{H}_\Lambda$ is actually $\mathcal{H}\Lambda$ for $\beta = \tau'\lambda$ and $b = -2\tau''b'$. Indeed, the explicit form above is obtained for the values of β and b just specified and substituting $\mathcal{P}$ and $\mathcal{Q}$ for P and Q in (3.13) accordance to (2.22), (2.23). When one restricts to $\mathfrak{L}_\Lambda$, Λ is no longer required on the right-hand side of (3.15). Then, relation (2.17) and theorem 2.2 imply that $\mathcal{H}_\Lambda$ admits the reflection algebra symmetry.

Integrability is proved upon expanding the Sklyanin determinant. One can show that it can be written as

$$\mathrm{sdet}(\mathcal{S}(u)\Lambda) = \Lambda + \sum_{k=0}^{+\infty} \frac{1}{u^{k+1}} \left[\lambda(n-1) \sum_{i=1}^{N} (1 + (-1)^k) d_i^k + G_k(d_1, \ldots, d_N) \right] \Lambda \tag{3.16}$$

where G_k is a N-variable polynomial of highest degree $k - 1$. We denote by $\mathcal{I}_k$ the term between brackets in (3.16). Since the coefficients of the Sklyanin determinant are central elements, one deduces that

$$[\mathcal{I}_k\Lambda, \mathcal{I}_l\Lambda] = 0 \qquad \text{and} \qquad [\mathcal{I}_k\Lambda, \mathcal{H}_\Lambda] = 0 \qquad \forall\, k, l \in \mathbb{Z}_{\geqslant 0} \tag{3.17}$$

and by paying attention to the terms of highest order in the partial derivatives in $\mathcal{I}_k\Lambda$, it is readily seen that $\{\mathcal{I}_{2k}\Lambda\}_{1\leqslant k\leqslant N}$ are independent, which proves the integrability. □

3.2. Physical Hamiltonians and gauge fixing

We still have to refine the form of the above Hamiltonian $\mathcal{H}_\Lambda$ so that its physical interpretation will be easier. The aim is to recover the usual physical Hamiltonian in units of $\hbar^2/2m$

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} + V(z_1, \ldots, z_N) \tag{3.18}$$

for some potential V. This can be achieved by performing a gauge transformation $\mu(\mathbf{q})$ and a change of variables $\mathbf{q} = \xi(\mathbf{z})$ with $\mathbf{q} = (q_1, \ldots, q_N)$, $\mathbf{z} = (z_1, \ldots, z_N)$

$$H = \mu(\mathbf{q})\mathcal{H}_\Lambda \frac{1}{\mu(\mathbf{q})}\bigg|_{\mathbf{q}=\xi(\mathbf{z})}. \tag{3.19}$$

We note that this does not affect the results about the symmetry and the integrability.

To obtain (3.18) from $\mathcal{H}_\Lambda$ given in (3.15), the suitable transformations are

$$\xi(\mathbf{z}) = (A^{-1}(\mathrm{i}z_1), \ldots, A^{-1}(\mathrm{i}z_N)) \tag{3.20}$$

$$\mu(\mathbf{q}) = \prod_{1\leqslant i\leqslant N} \sqrt{a(q_i)}. \tag{3.21}$$

Theorem 3.3. *Under the transformations (3.20), (3.21), the potential V in (3.18) splits into an external potential, U, and a spin potential, V_{spin},*

$$V(\mathbf{z}) = V_{\text{spin}}(\mathbf{z}) + \sum_{k=1}^{N} U(z_k) \tag{3.22}$$

with

$$\begin{aligned} V_{\text{spin}}(\mathbf{z}) = &- \sum_{1\leqslant i<j\leqslant N} \left(\frac{\gamma\lambda\left(P_{ij} - \frac{\lambda}{2\gamma}\right)}{\sin^2[\gamma(z_i - z_j)]} + \frac{\gamma\lambda\left(\overline{P}_{ij} - \frac{\lambda}{2\gamma}\right)}{\sin^2[\gamma(z_i + z_j)]} \right) \\ &+ \sum_{i=1}^{N} \left(\frac{\gamma(b'+c')\left(Q_i + \frac{b'+c'}{2\gamma}\right)}{2\sin^2(\gamma z_i)} + \frac{\gamma(b'-c')\left(Q_i + \frac{b'-c'}{2\gamma}\right)}{2\cos^2(\gamma z_i)} \right) \end{aligned} \tag{3.23}$$

and

$$U(x) = \tfrac{1}{4}a'(A^{-1}(\mathrm{i}x))^2 - \tfrac{1}{2}a(A^{-1}(\mathrm{i}x))a''(A^{-1}(\mathrm{i}x)) \tag{3.24}$$

where $a'(x) = \mathrm{d}a(x)/\mathrm{d}x$.

Proof. By direct computation □

To complete our discussion, we have to specify how the wavefunction and the relations (3.1), (3.3) transform under the change of variables (3.20). The wavefunction ϕ' on which H acts is given by

$$\phi'(z_1, \ldots, z_N|s_1, \ldots, s_N) = \phi(A^{-1}(\mathrm{i}z_1), \ldots, A^{-1}(\mathrm{i}z_N)|s_1, \ldots, s_N). \tag{3.25}$$

It is then straightforward to see that the action of $\mathcal{P}$ is unchanged

$$\mathcal{P}_{ij}\phi'(z_1, \ldots, z_i, \ldots, z_j, \ldots, z_N|s_1, \ldots, s_N) = \phi'(z_1, \ldots, z_j, \ldots, z_i, \ldots, z_N|s_1, \ldots, s_N)$$

and, noting that $\alpha(A^{-1}(\mathrm{i}z)) = A^{-1}(-\mathrm{i}z)$, the action of $\mathcal{Q}$ reads

$$\mathcal{Q}_i\phi'(z_1, \ldots, z_i, \ldots, z_N|s_1, \ldots, s_N) = \phi'(z_1, \ldots, -z_i, \ldots, z_N|s_1, \ldots, s_N) \tag{3.26}$$

i.e. it is independent of α when we work with the variables z_i. For wavefunctions in $\mathfrak{L}_\Lambda$, this implements the Neumann (resp. Dirichlet) boundary condition for $\tau'' = 1$ (resp. $\tau'' = -1$).

We can give some comments on the form of the potentials. The term $\gamma\lambda\left(P_{ij} - \frac{\lambda}{2\gamma}\right)/(\sin[\gamma(z_i - z_j)])^2$ expresses the usual two-body interaction between the ith and jth particles and does not break translation invariance as expected. The additional terms can be better interpreted if one imagines a 'mirror' sitting at the origin $z = 0$. Then, the term $\gamma\lambda\left(P_{ij}-\frac{\lambda}{2\gamma}\right)/(\sin[\gamma(z_i+z_j)])^2$ represents the two-body interaction between the ith particle and the 'mirror-image' of the jth particle. And the remaining terms involving only z_i accounts for the potential of the 'impurity' at the origin. These terms clearly violate translation invariance. Indeed, defining the total momentum as usual

$$\mathcal{I} = -\mathrm{i}\sum_{i=1}^{N} \frac{\partial}{\partial z_i} \tag{3.27}$$

it is readily seen that

$$[\mathcal{I}, H] \neq 0. \tag{3.28}$$

We want to emphasize that this interpretation in terms of an impurity sitting at the origin and of a 'mirror-image' of the system is possible thanks to (3.26), which is actually related to the fact that the Hamiltonian H is invariant under the space reflections $z_i \to -z_i, i = 1, \ldots, N$.

3.3. Examples

In all the above constructions, we have some freedom with the function a and the constants γ and c'. In this section, we use this freedom to exhibit particular Hamiltonians admitting the reflection algebra as symmetry algebra.

We work with the Hamiltonian (3.18) and from the previous section, we know that we control the external potential U thanks to a irrespective of V_{spin}. Thus, we suppose that the function a is constant so that the scalar external potential, U, vanishes.

3.3.1. B_N-type nonlinear Schrödinger Hamiltonian. Let

$$\gamma = \mathrm{i}\gamma' \qquad \lambda = \mathrm{i}g \qquad b' = -\mathrm{i}b_1 \qquad \text{where} \quad \gamma', g, b_1 \in \mathbb{R}. \tag{3.29}$$

Taking the limit $\gamma' \to +\infty$ in (3.23) in the sense of distributions, we obtain

$$H_{\text{NLS}} = -\sum_{k=1}^{N} \frac{\partial^2}{\partial z_k^2} + 2g \sum_{1\leqslant k<l\leqslant N} [\delta(z_k - z_l)P_{kl} + \delta(z_k + z_l)\overline{P}_{kl}] + 2b_1 \sum_{k=1}^{N} \delta(z_k)Q_k. \tag{3.30}$$

We know from the above results that this Hamiltonian admits the reflection algebra symmetry and is integrable. Let us note that when acting on $\mathfrak{L}_\Lambda$, we can drop the spin operators $P_{ij}, \overline{P}_{ij}, Q_i$ in this particular case due to the presence of the delta functions

$$H_{\text{NLS}} = -\sum_{k=1}^{N} \frac{\partial^2}{\partial z_k^2} + 2g\tau' \sum_{1\leqslant k<l\leqslant N} [\delta(z_k - z_l) + \delta(z_k + z_l)] + 2b_1\tau'' \sum_{k=1}^{N} \delta(z_k). \tag{3.31}$$

This is the Hamiltonian of a system of N bosonic ($\tau' = 1$) or fermionic ($\tau' = -1$) particles interacting through a delta potential with coupling constant g in the presence of a delta-type impurity sitting at the origin.

3.3.2. B_N trigonometric/hyperbolic Sutherland model with spin. To recover the known integrable Hamiltonian of the B_N trigonometric Sutherland model with spin [15], we take particular values of the constants present in (3.18)–(3.23)

$$\gamma = 1 \quad \lambda = 2g \quad b' + c' = -2b_1 \quad \text{and} \quad b' - c' = -2b_2 \quad \text{where}\, g, b_1, b_2 \in \mathbb{R}. \tag{3.32}$$

Thus, the Hamiltonian (3.18) becomes

$$\begin{aligned} H_{\text{BtS}} = &-\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} - 2g \sum_{1\leqslant i<j\leqslant N} \left(\frac{(P_{ij} - g)}{\sin^2(z_i - z_j)} + \frac{(\overline{P}_{ij} - g)}{\sin^2(z_i + z_j)} \right) \\ &- \sum_{i=1}^{N} \left(\frac{b_1(Q_i - b_1)}{\sin^2(z_i)} + \frac{b_2(Q_i - b_2)}{\cos^2(z_i)} \right). \end{aligned} \tag{3.33}$$

g is the coupling constant and b_1, b_2 parametrize the coupling with the impurity. From the general results of the previous sections, we know that the reflection algebra is the symmetry of the Hamiltonian (3.33).

The Hamiltonian of B_N hyperbolic Sutherland model with spin [12] is obtained by setting

$$\gamma = i \quad \lambda = 2ig \quad b' + c' = -2ib_1 \quad \text{and} \quad b' - c' = 2ib_2 \quad \text{where } g, b_1, b_2 \in \mathbb{R} \tag{3.34}$$

and it takes the same form as (3.33) but for the trigonometric functions replaced by the corresponding hyperbolic functions.

4. Hamiltonians with $Y(n)$ symmetry

In this section, we take advantage of theorem 2.3 and just adapt all our machinery to exhibit a general integrable Hamiltonian with Yangian symmetry which, once particularized, reproduces already known systems such as nonlinear Schrödinger and A_N Sutherland models with spin.

4.1. Representation of $\widetilde{\mathcal{A}}(N)$ and associated Hamiltonians

It is easy to see that $\sum_{i=1}^N d_i^2$ also appears in the expansion of $\operatorname{qdet}\mathcal{T}(u)$ in (2.13). As is customary in the literature [3, 16, 17], the starting point is a representation of the degenerate affine Hecke algebra, $\widetilde{\mathcal{A}}(N)$. We keep (3.1) and (3.2) and take for the Dunkl operator

$$d_l = a(q_l)\frac{\partial}{\partial q_l} + \sum_{k=1}^{l-1} v(q_l, q_k)\mathcal{P}_{kl} - \sum_{k=l+1}^{N} v(q_k, q_l)\mathcal{P}_{lk}. \tag{4.1}$$

At this stage, we can reproduce along the same line the arguments of section 3 to state the following theorems whose proofs are similar to that of theorems 3.1, 3.2 and will not be given here.

Theorem 4.1. *For $a \neq 0$ and $A(x) = \int^x \frac{\mathrm{d}y}{a(y)}$ invertible, the operators $\mathcal{P}_{ij}$ and d_i as defined in (3.1) and (4.1) realize $\widetilde{\mathcal{A}}(N)$ if and only if*

$$v(x, y) = \frac{\beta}{\mathrm{e}^{-2\gamma(A(x)-A(y))} - 1} \qquad \gamma \in \mathbb{C}. \tag{4.2}$$

Again, we can construct the effective Hamiltonian $\widetilde{\mathcal{H}}_{\Lambda^{(1)}}$ whose properties are gathered in:

Theorem 4.2. *When restricted to $\mathfrak{L}_{\tau'}^{(1)}$, the effective Hamiltonian*

$$\widetilde{\mathcal{H}}_{\Lambda^{(1)}} = \sum_{i=1}^N \left(a(q_i)^2 \frac{\partial^2}{\partial q_i^2} + a(q_i)\frac{\partial a(q_i)}{\partial q_i}\frac{\partial}{\partial q_i} \right) + \sum_{1\leqslant i<j\leqslant N} \left(\frac{\gamma\lambda\left(P_{ij} - \frac{\lambda}{2\gamma}\right)}{\sinh^2[\gamma(A(q_i) - A(q_j))]} \right) \tag{4.3}$$

admits the Yangian symmetry and is integrable.

Now, performing the transformations (3.20), (3.21) on $\widetilde{\mathcal{H}}_{\Lambda^{(1)}}$ we obtain the following physical Hamiltonian:

$$\tilde{H} = -\sum_{i=1}^N \frac{\partial^2}{\partial z_i^2} + \tilde{V}_{\text{spin}}(\mathbf{z}) + \sum_{i=1}^N U(z_i) \tag{4.4}$$

with U given in (3.24) and

$$\tilde{V}_{\text{spin}}(\mathbf{z}) = -\sum_{1\leqslant i<j\leqslant N} \frac{\gamma\lambda\left(P_{ij} - \frac{\lambda}{2\gamma}\right)}{\sin^2[\gamma(z_i - z_j)]}. \tag{4.5}$$

Remark. In the expansion of qdet $\mathcal{T}(u)$ in (2.13), it is easy to see that there appears the operator

$$\sum_{i=1}^{N} d_i = \sum_{i=1}^{N} a(q_i)\frac{\partial}{\partial q_i}. \tag{4.6}$$

Assuming that a is constant and performing the transformations (3.20), (3.21), (4.6) becomes $\mathcal{I}$ given in (3.27). We then conclude that $\mathcal{I}$ commutes with our general Hamiltonian $\widetilde{H}$ so that the system is translation invariant. In particular, this shows that the systems we will consider in the following section with Yangian symmetry are translation invariant as expected.

4.2. Examples

Using the freedom on a and γ in exactly the same fashion as in section 3.3, we show that the Hamiltonian (4.4) generalizes known Hamiltonians for which the Yangian symmetry and the integrability had already been proved:

- Nonlinear Schrödinger Hamiltonian [3] ($\gamma = \mathrm{i}\gamma'$, $\lambda = \mathrm{i}g$, $\gamma', g \in \mathbb{R}$, $\gamma' \to +\infty$)

$$\widetilde{H}_{\mathrm{NLS}} = -\sum_{k=1}^{N} \frac{\partial^2}{\partial z_k^2} + 2g\tau' \sum_{1\leqslant k<l\leqslant N} \delta(z_k - z_l). \tag{4.7}$$

- A_N trigonometric Sutherland model with spin [18, 19] ($\gamma = 1$, $\lambda = 2g$, $g \in \mathbb{R}$)

$$\widetilde{H}_{AtS} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} - 2g \sum_{1\leqslant i<j\leqslant N} \left(\frac{(P_{ij} - g)}{\sin^2(z_i - z_j)}\right). \tag{4.8}$$

- A_N hyperbolic Sutherland model with spin [18, 19] ($\gamma = \mathrm{i}$, $\lambda = 2\mathrm{i}g$, $g \in \mathbb{R}$)

$$\widetilde{H}_{AhS} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} - 2g \sum_{1\leqslant i<j\leqslant N} \left(\frac{(P_{ij} - g)}{\sinh^2(z_i - z_j)}\right). \tag{4.9}$$

5. Conclusion and outlooks

Starting from a representation of the *extended* degenerate affine Hecke algebra in terms of operators acting on wavefunctions, our main results are the construction of a general N-particle Hamiltonian and the proof that it admits the reflection algebra symmetry (theorems 2.2 and 3.2). This ensures in particular its integrability. The Yangian counterpart of this procedure gives back well-known results.

The physical investigation of this Hamiltonian shows that it is invariant under space reflections so that we considered wavefunctions whose behaviour under the action of the operator $\mathcal{Q}_i$ is dictated by a parameter $\tau'' = \pm 1$. This amounts to encoding a Neumann or Dirichlet boundary condition at $z = 0$. However, one sees that the 'mirror-image' of the system on the half-line is relevant and cannot be neglected if one wants to maintain the usual nontrivial two-body interactions. Of course, all this applies to the already known systems to which our general Hamiltonian reduces in appropriate limits.

This brings us to the interesting issue of diagonalizing the Hamiltonian H using available results for reflection algebras. This would provide the spectrum for apparently distinguished models (such as B_N-type NLS or B_N trigonometric/hyperbolic Sutherland models), with boundary, unified by the Hamiltonian H.

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